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Best Approximation by Monotone Functions

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I. INTRODUCTION

For $1 \leqslant p < \infty$, let L_p denote the Banach space of pth power Lebesgue integrable functions on the interval $[0, 1]$ with $\|f\|_p = (\int_0^1 |f|^p)^{1/p}$. Let $M_p \subseteq L_p$ denote the set of non-decreasing functions. Then M_p is a closed convex lattice. For $1 < p < \infty$, each $f \in L_p$ has a unique best approximation from M_p, while, for $p = 1$, existence of a best approximation from M_p follows from Proposition 4 of [6].

Recently, there has been interest in characterizing best L_1 approximations from M_1 [1, 2, 3, 4]. For example, in [1] it is shown that if $f \in L_\infty$ and if each point in $[0, 1]$ is a Lebesgue point of f [7], then the best L_1 approximation to f from M_1 is unique and continuous. In each of the papers mentioned above, the approach taken was measure theoretic, and the arguments were necessarily complicated.

The purpose of this paper is to approach the best approximation problem from a duality viewpoint. This leads to considerable simplification in the derivation of the results, and allows for the omission of the assumption that $f \in L_\infty$.

II. PRELIMINARIES

For $1 < p < \infty$, it is known [5] that $g^* \in M_p$ is the best approximation to $f \in L_p$ if and only if

$$\int_0^1 (g^* - g)(f - g^*)\,|f - g^*|^{p-2} \geqslant 0 \tag{2.1}$$

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for all $g \in M_p$. For $p=1$, $g^* \in M_1$ is a best approximation to f if and only if there exist $\phi \in L_\infty$, $\|\phi\|_\infty = 1$, much that

$$\int_0^1 \phi(f-g^*) = \|f-g^*\|_1 \tag{2.2}$$

and

$$\int_0^1 \phi(g^*-g) \geqslant 0 \tag{2.3}$$

for all $g \in M_1$. In addition, if $f(x) \neq g^*(x)$, then $\phi(x) = \text{sign}\,(f(x) - g^*(x))$.

Define

$$\phi_p = \begin{cases} (f-g^*)\,|f-g^*|^{p-2}, & 1<p<\infty \\ \phi, & p=1 \end{cases} \tag{2.4}$$

and define, for $0 \leqslant a \leqslant 1$,

$$r_p(a) = \int_0^a \phi_p. \tag{2.5}$$

LEMMA 1. *The following properties hold*:

(i) $\int_0^1 g^*\phi_p = 0$;

(ii) $r_p(a) \geqslant 0$;

(iii) $\int_0^1 \phi_p = 0$;

(iv) $\int_a^1 \phi_p \leqslant 0$;

(v) *If* g^* *jumps at* $a \in (0,1)$, *then* $r_p(a) = 0$; *and*

(vi) *If* $r_p(a) > 0$, *then* g^* *is constant in a neighborhood of* a $(a \in (0,1))$.

Proof. (i) From (2.1) and (2.3),

$$\int_0^1 g^*\phi_p \geqslant \int_0^1 g\phi_p \tag{2.6}$$

for all $g \in M_p$. Choosing $g = 2g^*$ yields

$$\int_0^1 g^*\phi_p \leqslant 0$$

whil choosing $g = \frac{1}{2}g^*$ yields

$$\int_0^1 g^*\phi_p \geqslant 0.$$

(ii) The proof follows from (2.6) and (i) by choosing

$$g = \begin{cases} -1, & 0 \leqslant x \leqslant a \\ 0, & a < x \leqslant 1 \end{cases}$$

(iii) The proof follows from (2.6) and (i) by alternately choosing

$$g \equiv 1 \qquad \text{and} \qquad g \equiv -1.$$

(iv) follows from (ii) and (iii).

(v) Choose $\varepsilon > 0$ so that $\varepsilon < a < 1 - \varepsilon$. Let $g = g^*$ for $0 \leqslant x \leqslant \varepsilon$ and for $1 - \varepsilon \leqslant x \leqslant 1$. Then, from (2.6), (i) and the boundedness of g^*, g on $[\varepsilon, 1 - \varepsilon]$, we have

$$0 \leqslant \int_{\varepsilon}^{1-\varepsilon} r_p \, dg^* \leqslant \int_{\varepsilon}^{1-\varepsilon} r_p \, dg.$$

Suppose $r_p(\bar{a}) = 0$ for some $\bar{a} \in (\varepsilon, 1 - \varepsilon)$. Let

$$g(x) = \begin{cases} g^*(\varepsilon), \; \varepsilon \leqslant x < \bar{a} \\ g^*(1 - \varepsilon), \; \bar{a} < x \leqslant 1 - \varepsilon \end{cases}$$

Then $\int_{\varepsilon}^{1-\varepsilon} r_p \, dg = 0$. But if $r_p(a) > 0$, then $\int_{\varepsilon}^{1-\varepsilon} r_p \, dg^* \geqslant r_p(a)(g^*(a^+) - g^{*\prime}(a^-)) > 0$.

Contradiction.

If $r_p(a) > 0$ for all a, $0 < a < 1$, then there is a sequence $\varepsilon_n \to 0$ such that r_p, on $[\varepsilon_n, 1 - \varepsilon_n]$, takes on its min at ε_n or $1 - \varepsilon_n$. Suppose the min is taken on at ε_n. Let $g(x) = g^*(1 - \varepsilon_n)$, $\varepsilon_n < x < 1 - \varepsilon_n$. Then $\int_{\varepsilon_n}^{1-\varepsilon} r_p \, dg = r_p(\varepsilon_n)(g^*(1 - \varepsilon_n) - g^*(\varepsilon_n))$ and

$$\int_{\varepsilon_n}^{1-\varepsilon_n} r_p \, dg^* \geqslant r_p(\varepsilon_n)(g^*(1 - \varepsilon_n) - g^*(\varepsilon_n)).$$

Hence g^* can only jump at ε_n. A similar argument applies if r_p takes on its min at $1 - \varepsilon_n$. Letting $n \to \infty$, we see that g^* cannot jumpt at any a, $0 < a < 1$.

(vi) The proof if (v) shows that if $r_p(a) > 0$, then g^* is continuous at a.

Let $\varepsilon > 0$ be sufficiently small so that $\varepsilon < a < 1 - \varepsilon$. Suppose $r_p(\bar{a}) = 0$ for some $\bar{a} \in (\varepsilon, 1 - \varepsilon)$. Let

$$g(x) = \begin{cases} g^*(\varepsilon), \; \varepsilon \leqslant x < \bar{a} \\ g^*(1 - \varepsilon), \; \bar{a} < x \leqslant 1 - \varepsilon. \end{cases}$$

Then $\int_\varepsilon^{1-\varepsilon} r_p\, dg = 0$. By the continuity of r_p, there exists $x_1 < a < x_2$ such that $\min_{x_1 \leqslant x \leqslant x_2} r_p(x) > 0$. Consequently

$$\int_\varepsilon^{1-\varepsilon} r_p\, dg^* \geqslant (\min_{x_1 \leqslant x \leqslant x_2} r_p(x))(g^*(x_2) - g^*(x_1)).$$

Hence $g^*(x_2) = g^*(x_1)$.

If $r_p(a) > 0$ for all a, $0 < a < 1$, there is a sequence $\varepsilon_n \to 0$ so that r_p is non-constant on $[\varepsilon_n, 1-\varepsilon_n]$ and $\min_{\varepsilon_n \leqslant x \leqslant 1-\varepsilon_n} r_p(x) = \min\{r_p(\varepsilon_n), r_p(1-\varepsilon_n)\}$. Then, as in (v),

$$\int_{\varepsilon_n}^{1-\varepsilon_n} r_p\, dg^* = \min\{r_p(\varepsilon_n), r_p(1-\varepsilon_n)\}\ (g^*(1-\varepsilon_n) - g^*(\varepsilon_n)).$$

On the other hand, it is easy to see that if r_p is non-constant and g^* is non-constant, then $\int_{\varepsilon_n}^{1-\varepsilon_n} r_p\, dg^* = \min\{r_p(\varepsilon_n), r_p(1-\varepsilon_n)\}\ (g^*(1-\varepsilon_n) - g^*(\varepsilon_n))$. Hence g^* is constant on $[\varepsilon_n, 1-\varepsilon_n]$. Letting $n \to \infty$, we see that g^* is constant on $(0, 1)$.

III. Main Results

In this section we establish, under mild assumptions on f, continuity of the best approximation to $f \in L_p$ from M_p, and for $p = 1$, unicity.

If A is a measurable subset of $[0, 1]$ and I is a subinterval of $[0, 1]$, define the upper metric density of A at x by

$$\bar{m}(A, x) = \lim_{n \to \infty} \sup_I \{m(A \cap I)/mI;\ x \in I,\ mI < 1/n\}.$$

The lower metric density of A at x, $\underline{m}(A, x)$, is defined similarly with sup replaced by inf. The metric density, $m(A, x)$, is $m(A, x) = \bar{m}(A, x) = \underline{m}(A, x)$ if equality holds. x is a Lebesgue point [7] of f if and only if, for each $\varepsilon > 0$,

$$A_\varepsilon = \{y;\ |f(y) - f(x)| < \varepsilon\}$$

has metric density one at x.

THEOREM 1. *Suppose $f \in L_p$, $1 \leqslant p < \infty$, and $g^* \in M_p$ is a best approximation to f. If x_0 is a Lebesgue point of f, then g^* is continuous at x_0.*

Proof. We provide the details for $1 < p$. The case $p = 1$ is similar.

From Lemma 1, we need only consider the case $r_p(x_0) = 0$.

Suppose $f(x) < g^*(x_0^+)$.
Select $\delta > 0$ so that $f(x_0) + \delta < g^*(x_0^+)$.
Then

$$\int_{A_\delta \cap (x_0, x_0+\varepsilon)} (f - g^*) \mid f - g^* \mid^{p-2}$$

$$< (f(x_0) + \delta - g^*(x_0^+)) \mid f(x_0) + \delta - g^*(x_0^+) \mid^{p-2} m(A_\delta \cap (x_0, x_0 + \varepsilon))$$

Since

$$\lim_{\varepsilon \to 0} \frac{m(A_\delta^c \cap (x_0, x_0 + \varepsilon))}{\varepsilon} = 0,$$

we must have

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{A_\delta^c \cap (x_0, x_0+\varepsilon)} (f - g^*) \mid f - g^* \mid^{p-2} = 0$$

It follows that, for sufficiently small ε,

$$\int_{x_0}^{x_0+\varepsilon} (f - g^*) \mid f - g^*) \mid^{p-2} < 0.$$

However $r_p(x_0) = 0$ and Lemma 1 (ii) imply

$$0 \leqslant \int_0^{x_0+\varepsilon} (f - g^*) \mid f - g^*) \mid^{p-2} = \int_{x_0}^{x_0+\varepsilon} (f - g^*) \mid f - g^* \mid^{p-2}$$

This contradiction establishes that $g^*(x_0^+) \leqslant f(x_0)$.
In a similar fashion, it can be shown that $f(x_0) \leqslant g^*(x_0^-)$.
Hence g^* is continuous at x_0.

Remark. If f is continuous on $[0, 1]$, then so is g^*.

THEOREM 2. *Let $f \in L_1$ and suppose that every point in* $(0, 1)$ *is a Lebesgue point of f. Then the best approximation to f from M_1 is unique on* $(0, 1)$.

Proof. Let g_1, g_2 be two best approximations to f. The inequality

$$\mid f(x) - \tfrac{1}{2}(g_1(x) + g_2(x)) \mid \leqslant \tfrac{1}{2} \mid f(x) - g_1(x) \mid + \tfrac{1}{2} \mid f(x) - g_2(x)) \mid$$

together with

$$\| f - \tfrac{1}{2}(g_1 + g_2) \|_1 = \tfrac{1}{2} \| f - g_1 \|_1 + \tfrac{1}{2} \| f - g_2 \|_1$$

shows that, for almost all x, if $f(x) \leqslant g_1(x)$, then $f(x) \leqslant g_2(x)$ and if $f(x) \geqslant g_1(x)$, then $f(x) \geqslant g_2(x)$.

Let

$$\Omega_1 = \{x; f(x) > g(x)\}$$

$$\Omega_2 = \{x; f(x) < g(x)\}$$

and

$$\Omega_3 = \{x; f(x) = g(x)\}$$

where $g = \frac{1}{2}(g_1 + g_2)$. On Ω_3, $f = g_1 = g_2$.

Let $x_0 \in \Omega_1$. Choose $\delta > 0$ so that $f(x_0) > g(x_0) + \delta$. By the continuity of g there is $\varepsilon > 0$ so that $f(x_0) > g(x) + \delta$ for all $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$. If $r_1(x_0) = 0$, then $\int_{x_0}^{x_0+\varepsilon} \phi_1 > 0$ since $\phi_1 = 1$ on $A_\delta \cap (x_0, x_0 + \varepsilon)$ and A_δ has metric density one. Similarly $\int_{x_0-\varepsilon}^{x_0} \phi_1 > 0$. But this contradicts $\int_{x_0-\varepsilon}^{x_0} \phi_1 \leqslant 0$. Therefore $r_1(x_0) > 0$. Then, by Lemma 1 (vi), g is constant in a neighborhood of x_0. Hence g_1 and g_2 are constant in a neighborhood of x_0. Similarly, g_1, g_2 are constant in a neighborhood of each point of Ω_2. Since $f = g_1 = g_2$ on Ω_3, then, by the continuity of g_1 and g_2, we have $g_1 = g_2$ on $\Omega_1 \cup \Omega_2$. Hence, the best approximation to f is unique on (0, 1).

Remark 1. There is no need to assume $f \in L_\infty$ as in [1].

Remark 2. If $f \in L_\infty$, then existence of a best non-decreasing approximant follows from the fact that the set of non-decreasing functions in L_∞ is weak star closed. Perhaps the approach could have been used in [8].

Note added in proof. It has been pointed out by the referee that "Lebesgue point" should be replaced by "point of approximate continuity."

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